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Squig Sheets and Some Other Squig Fractal Constructions[1]

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Squig intervals are a class of hierarchically constructed fractals introduced by the author. They can be visualized as the final outcome upon a straight interval of a suitable cascade of local perturbative "eddies" ruled by two processes called decimation and separation. Their theory is summarized and their scope is extended in several new directions, especially by introducing new forms of separation. Squig intervals are generalized in two dimensions, with fractal dimensions ranging from 1.2886 to 1.589. Squig sheets are constructed in three-dimensional space with fractal dimensions ranging from 8/3 up. They should prove useful in modeling the fractal surfaces associated with turbulence and related phenomena. Squig intervals are constructed in three dimensions. Nonsymmetric "eddies" and the resulting squigs are tackled. Squig trees and intervals are drawn on unconventional lattices, either in the plane or in a prescribed fractal surface. Peyrière's M systems are mentioned: their study includes the proof that the informal "renormalization" argument (involving a transfer matrix) is exact for squigs.

KEY WORDS: Fractals; squig models.

1.3. INTRODUCTION

Squigs are a family of random fractals that I conceived of in 1978[1]; the theory of squig intervals, which are self-avoiding squig curves in the plane, is sketched in Chapter 24 of my 1982 book[2] (see footnote 3), to be referred to

[1] Presented at the *Third Conference on Fractals: Fractals in the Physical Sciences*, held at the National Bureau of Standards, Gaithersburg, Maryland, on November 20–23, 1983.

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[3] The reader's attention should be drawn to the fact that the second and later printings of this book include an update chapter and additional references. Though it should not have been necessary, it may be useful also to mention here that most of the material in this book that concerns physics, e.g., polymers and percolation clusters, was *not* found in either of my two earlier Essays on fractals, *Les objects fractals: forme, hasard et dimension* (Flammarion,

as FGN. There is a growing literature on the squigs' geometry, their physics, and their applications.[1-9] The purpose of the present paper is to generalize an essential ingredient of the construction of squig curves in the plane, the notion of separation, and to introduce and investigate squig curves and sheets in spaces of more than two dimensions, and in unconventional underlying lattices.

These examples are not meant to be exhaustive, but to demonstrate the versatility of the concept of squig. Many other examples come to mind, and squigs can be expected to become widely used on a substitute to the Koch curves, which had been vital to the early history of fractal geometry and of its applications.

The present paper can be viewed as a direct continuation of part I of my Edingburgh *StatPhys 15* presentation.[3] This is why this paper's section numbers start with 1.3 and continue after those of Ref. 3. However, this is a free-standing paper, the basic problems, definitions, and procedures are restated in detail, and figures are inserted. In fact, it may be easier to study part I of Ref. 3 after the present work.[4]

The techniques used in this paper are (without this having been intended) examples of renormalization analysis, and obtain the fractal dimensionality via the averaged transfer matrix of these fractals (TMF).[10] J. Peyrière[7] has showed the results to be mathematically exact under wide conditions of validity; his analysis is not very accessible, but he has verified (private communication) that the results in Ref. 3 and in the present paper are exact. In particular, he has proved that a squig's Hausdorff–Besicovitch dimensionality coincides with the fractal dimensionality, derived by transfer matrix methods.[5]

Paris, 1975), and *Fractals: Form, Chance and Dimension* (W. H. Freeman, San Francisco, 1977).

[4] Part II of this Edinburgh paper is amplified in a *Comment* in *J. Stat. Phys.*, this issue.

[5] The full treatment (Ref. 7b) is lengthy and solely addressed to mathematicians. I hope that Peyrière will soon provide an English summary for physicists and general readers. His report, titled "Mandelbrot random bead strings and birth processes with interaction," dated 1978, is no longer available.

[6] A variant of the simplest squig was devised in an 1983 paper by D. J. Klein and W. A. Seitz (Ref. 8). The list of possible configurations is the same as in my 1978 construction as shown in Fig. 2. However, they are assigned a different set of probabilities. My model distinguishes the four configurations in Fig. 1, and gives each a probability equal to $1/4$. Klein and Seitz identify configurations 2 and 3, and give each of their three distinct configurations a probability equal to $1/3$. *A priori*, there was no imperative reason to prefer one probability assignment to the other. *A posteriori*, however, the renormalization analysis that Klein and Seitz require turns out to be lengthy, while only approximate, and it does not yield D itself only the bound $D \leqslant 1/.834499 = 1.1983$, which is very undesirably far from Flory's value $4/3$.

Nomenclature. As recommended in Ref. 3, the term "Hausdorff–Besicovitch dimensionality" is now reserved to cases where the algorithm due to Hausdorff and Besicovitch was actually implemented. This algorithm is of wide applicability but difficult. The generic notion which it is meant to measure is now denoted as "fractal dimensionality." FGN followed a less formal course, and *Hausdorff–Besicovitch* and *fractal* were often made synonymous. This old informality has now become untenable. For other reasons why *fractal* has broken off from *Hausdorff–Besicovitch*, see Ref. 3, Section 2.

1.3.1. Historical Development of the Notion of Fractal Squig, from Intervals, to Trees Imbedded in a Given Surface, to Clusters, and on to Sheets and Beyond

The earliest squigs were "intervals" and "trees" in triangular lattices of base $b = 2$, whose construction is recalled again in Section 1.3.3. They were meant, respectively, to be models of linear polymers and of river networks. The match proves surprisingly excellent. In particular, the fractal dimensionality D is close to the dimensionality 4/3 that is known to characterize the self-avoiding random walk. This value is also excellent to describe linear polymers. Squigs were intended to be non-self-intersecting, and the proof that this intent was fulfilled is found in Ref. 9, p. 384. (Aside: $D = 4/3$ is also not unreasonable for rivers—though on the high side. In addition, squig trees of base $b = 2$ satisfy Horton's law, which is an important fact about rivers, but on which we cannot dwell here.)

Many fractal curves, e.g., the simple Koch snowflake, can be thought of as the final outcome of the action upon a straight interval of a hierarchical cascade of random "eddies" of decreasing size. In the case of squigs, the underlying process bears some resemblance to a physical vortex through a fluid. As is well known, it follows from the equations of motion that in time a fluid vortex becomes increasingly thin and long, hence very convoluted. In the squig constructions, we imagine that this process occurs via a random hierarchy of eddies: each stage starts with the effects of larger eddies; it only adds detail. The squig construction also postulates that at each stage the vortex is of uniform thickness.

It will be seen that, on triangular planar lattices of base $b = 2$, the construction of squig trees and intervals involves a single operation, called "decimation." On other lattices, however, e.g., on square planar lattices of base $b = 2$, a second operation is needed, which can take several forms but will always be called "separation". "Bond separation" is the original method used in Refs. 1, 2, 3, 7, and 9. "Site separation" was briefly mentioned in Ref. 6, and will now be "retrofitted" to generate alternative squigs on a square lattice of base $b = 2$. Further forms of separation are studied here for the first time.

After the trees and the intervals, attention moved on to squig clusters, which are a new family of random fractals, first described in the Edingburgh paper,[3] and devised to match the properties of the two-dimensional percolation clusters at criticality. Again, as seen in Section 1.1 of the Edinburgh paper and in Ref. 5, the match proves surprisingly excellent. It extends to the fractal dimensionalities of the whole cluster and of every one of its parts that was studied. The branching properties also match. The order of ramification is designed to be finite but >2, and the dimensionality of recurrence $D_{\text{recurrence}}$ is found to be $\sim 1/3$, hence the fraction/spectral dimensionality $D_{\text{fracton}} = 2(1 - D_{\text{recurrence}})$ is $\sim 4/3$. This value is worth noting because $D_{\text{recurrence}}$ is an intrinsic summary of the practically useful branching characteristics, and because for most nonrandom fractals, $D_{\text{recurrence}}$ differs from 1/3. In addition, Ref. 5 describes some variants of the squig clusters: partly random clusters (based on nonrandom decimation and random bond separation) and nonrandom clusters (based on nonrandom decimation and introducing site separation is a nonrandom form).

Now that the Sierpinski gasket has yielded many of its secrets, and has shown its limitations, squigs should find many uses in physics. To help along, several further squigs are described in the present paper. Most novel and especially promising are squig "sheets" (Section 1.5.1) on a cubic lattice. Intuitively, they are the ultimate effects upon a square of a suitable hierarchical cascade of eddy perturbations in space. They are offered as possible models of interfaces in chemistry or in the study of turbulence.

1.3.2. Perspective: Physics in a Continuum or on a Lattice, and Hierarchical Physics on a Pertiling Lattice

To motivate interest in squigs, they should be placed in perspective, and in particular should be compared to earlier models for the same phenomena. Natural space is continuous, but much of statistical physics is carried out on a lattice, as exemplified by the self-avoiding walk model of real linear polymers or to the Broadbent–Hammersley model of real percolation. Still, the lattice model are hard mathematically and hard to simulate on the computer. Hierarchical models are well known to be far easier to work with. One hopes that they are equally (or near equally) acceptable as models of what really happens in real space.

Squig fractals are hierarchical. They can be drawn either by extrapolation or by interpolation. Since the results are equivalent, this paper describes each individual case via the process that comes most easily. In terms of extrapolation, recursivity is built into the squig construction by starting with a lattice whose cells collect in supercells which are up-sized versions of the cells. In terms of interpolation, the basic ingredient of the squig construction is a tiling in which each tile is covered *ad infinitum* by smaller tiles similar to the original one. Different subtiles of the same tile need not be of the same size. Tiles may be squares, triangles, etc., but may also be bounded by fractal surves (Section 1.7). In the terminology of FGN, they form a *pertiling*. The Latin prefix "per" means "thoroughly"; e.g., "perfect" originally meant "done thoroughly," "perforation" is a "hole through."

Squigs are not drawn on the sides of the cells, but on the dual lattice of "potential bonds," obtained by linking the centers of original cells that share a side. For example, when the lattice is triangular or square, these bonds form a hexagonal or square lattice. Some bonds are then deleted at random, in recursive fashion, using "decimation" and "separation." The remaining bonds are called "activated." The construction is tractable because the processes of decimation and separation are either nonrandom, or random but statistically independent.

1.3.3. Background: Triangular Lattice of Base $b = 2$, and the Process of Pure Decimation. The Prototypical Plane-Filling Squig Tree

Let us use the extrapolative construction. The first stage takes b^2 triangular lattice cells $C(0)$ that fit into a twice larger triangle $C(1)$, and activates all the b^2 bonds between neighboring $C(0)$'s. The result is a small Y. The second construction stage takes b^2 copies of $C(1)$ that fit together in a b times larger triangle $C(2)$. The boundary between any two neighboring $C(1)$ within $C(2)$ is crossed by b potential bonds; one of them is picked at random and activated and the remaining $b - 1$ are decimated, yielding a tree. The kth stage takes b^2 statistically independent replicas of $C(k-1)$ to make a b larger triangle $C(k)$. The boundary between any two neighboring $C(k-1)$ is crossed by b^{k-1} potential bonds; one of them is activated, and the other $b^{k-1} - 1$ are decimated. The final outcome is a "squig tree" that fills the plane, hence is dimensionality $D = 2$.

Squig intervals. The direct (non-self-overlapping) path between two points in a squig tree is called a *squig interval.* By the term "interval from A to B," we denote a curve that does not self-intersect: it is a bicontinuous map of the straight interval $[0, 1]$. All the intervals containing A and B are curves, hence "connected," and they are "minimal." Connectedness means that any point P' on a curve may be displaced continuously until it coincides with any other point P'', and minimality means that removal of any portion

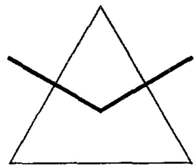

Fig. 1

of an interval from A to B (not including either A or B) necessarily destroys connectedness.

Now let me construct a squig interval without constructing a whole tree, via interpolation. Initially, we are given a triangle and the information that our curve crosses if from left to right between its upper sides. The first-order approximation, shown in Fig. 1, is an "elementary broken line" made of two segments meeting at the triangle's center (plus its bonds to the outside). Then we toss fair coins to determine which halves of the two upper sides are to be joined. The initial triangle is divided into equal fourths and we impose the condition that each side bounding a subtriangle can be crossed at most once. This implies that every triangle is crossed at most once. We are left with the four possibilities shown in Fig. 2a, b, c, and d, each having the probability 1/4. In the second-order approximation, the number of elementary broken lines is either 1 or 3. Selected examples of third-order approximations are shown in Fig. 3. A broken line approximation to the squig interval is made of intervals that join successive the centers of successive triangles. After this broken line has been oriented from one end to the other, each of its triangles can be classified as being either left or right handed, according to whether the break that the line suffers at this triangle's center bears to the left or to the right. The resulting sequence of letter R and L determines the approximation fully. To build the next broken line approximation, it suffices to determine whether it will cross each of the intertriangle boundaries through its left or its right half. See also footnote 6, p. 520.

Let us now bring in a useful "birth process" terminology. In the passage from one approximation to the next, one can say that each "mother triangle" "gives birth" to $N(1)$ smaller "daughter triangles," where $N(1)=1$ or $N(1)=3$ with the probabilities of 1/4 and 3/4, hence $\langle N(1)\rangle = 2.5$. If the

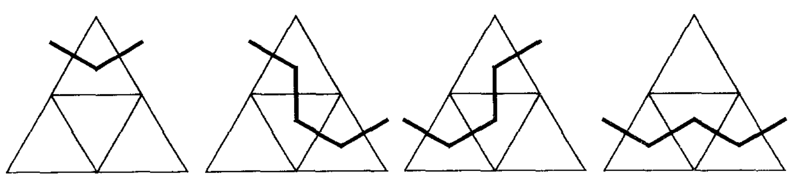

Fig. 2

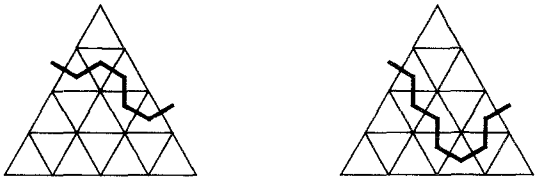

Fig. 3

births occurring in different triangles were statistically independent, we would deal with a process, known to probabilists, called "simple" birth process. The number $N(k)$ of sisters and cousins after k generations would satisfy $\langle N(k)\rangle = \langle N(1)\rangle^k = (2.5)^k$, and $N(k)$ divided by $(2.5)^k$ would converge to a limit random variable. This suggests the similarity dimension $D = \log_2 2.5 = 1.3219$. Actually, the hypothesis of independence of the births is invalid, nevertheless (see Section 1.8 or Ref. 7) the result is exact. By construction, the limit fractal squig does not self-intersect. It has also been shown (Ref. 9, p. 384) that, as intended, it does not self-contact either.

1.4. SQUIG INTERVALS IN THE PLANE CONSTRUCTED ON A SQUARE LATTICE OF BASE $b = 2$. ALTERNATIVE FORMS OF THE PROCESS OF SEPARATION

1.4.1. Squig Loops

The procedure that yielded trees in Section 1.3.3 can be implemented in other lattices, with varying outcomes. In some cases (e.g., Section 1.5) it yields alternative squig trees. In other cases, it yields fractals containing rings or loops.

Example: Squig loop structures on a square lattice of base $n = 2$. The first construction stage takes four square lattice cells that fit into a twice larger square. It is clear that the four bonds between neighboring $C(0)$'s form a small ring, and each successive construction stage will add further rings.

If one wants the procedure to yield trees and intervals, one needs an additional procedure, to be called "separation." This is a first complication. Further complication: even if each side can be crossed at most once, it is conceivable in the case of square lattices that a lattice cell be crossed twice. Section 1.4.2 will study a first form of separation, called "bond separation," which only allows a single transit, Section 1.4.3 will inject "site separation" which allows double transit, and Section 1.4.4 concerns other separations.

1.4.2. Random Bond Separation, and Single-Transit Squig Trees and Intervals

Squig trees. Go back to the elementary squig loops described above. Choose one of these bonds at random to be broken up or separated. This results in three activated bonds forming a small broken ring. The second construction stage fits four replicas of $C(1)$ in a larger square $C(2)$. Again, one of the four interfaces between the $C(1)$, chosen at random, is separated to prevent a super-ring from forming. The nonseparated interfaces are each crossed by two bonds which are decimated except for one. At the kth construction stage each nonseparated interface is crossed by 2^k potential bonds, which are decimated, except for one.

Squig intervals constrained to cross any cell in the pertiling once. To determine their structure and their dimensionality requires methods that are less direct than those used in Section 1.3.3. The reason is that the interval's transit through any given cell in the pertiling can occur either between neighboring sides (NS) or between opposite sides (OS). Examples of interpolations (analogous to Fig. 2) are shown in Fig. 4. Altogether, the passage from one to the next stage of interpolation in a squig interval is characterized by a matrix that was used without name in Refs. 1 and 7, and has recently been labeled the "transfer matrix of a fractal" (TMF).[10] Here, the TMF is a 2×2 matrix. Averaging all the separator's configurations yields an expected TMF, which is

	(NS)	(OS)
(NS)	5/4	3/4
(OS)	6/4	6/4

To obtain the fractal dimensionality of a fractal whose TMF is a $b \times b$ matrix, one seeks the leading eigenvalue λ_1, and one forms $D = \log_b \lambda_1$. Here, $\lambda_1 = 2.4430$ and $b = 2$, hence $D = \log_2 \lambda_1 = 1.2886$.

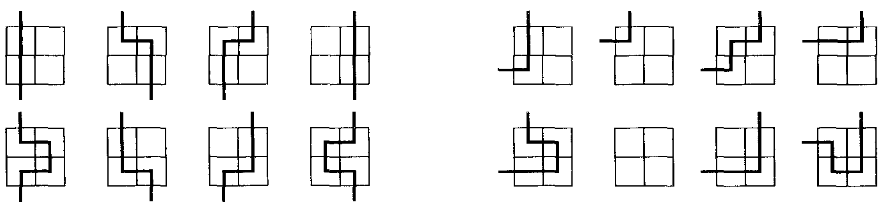

Fig. 4

1.4.3. Random Site Separation, and Double-Transit Squig Intervals

The alternative form of separation we now investigate, "site separation," increases D, from $D = 1.2886$ to nearly $D = \log_2 3 \sim 1.5849$.

Comment. Nearly the same range will be encountered in Section 1.6.1. This may be a numerical coincidence. But this may also reveal a deeper reality: it may be that this is a "natural range" for fractal curves that do not self-intersect, but have approximations that nearly self-contact.

To motivate site separation, we start with the interpolative construction of squig intervals, and take a closer look at the squig loops considered early in Section 1.4.1. The two nondecimated bonds that touch a given loop can attach either at different sites or at the same site. When interpolating a NS cell, they have the probability $1/16$ of being attached at the same site, which is the end point of four bonds. Bond separation, as performed in Section 1.4.2, continues to be implementable, and it completely erases the loop. In all cases, each cell is transited once by our approximate squig, hence also by the limit squig.

Now we change the rules, to allow a cell to be transited twice, say, from the top to the right then from the bottom to the left. Self-crossing or self-overlap remain prohibited, but we allow an approximate squig interval to self-contact. (The formal definition of a squig interval must be generalized, but the physicist is not worried.) Our list of cell types is thus extended to four possibilities: single transit between neighboring sides (NS), single transit between opposite sides (OS), double transit (DT) by the same interval, and perturbed transit (PT), which means single transit through a cell that is also being transited by a second unrelated squig interval. Starting with the configuration (NS), which allows for the situations where a loop is attached to two bonds, bond separation places a random diagonal "mirror" across this site: a self-contacting meander will be created with the probability s, and full loop in contact with an interval will be created with the probability $1 - s$. The quantity s will be called "seeding probability," because it "seeds" a (DT) configuration.

How to further interpolate the (DT) configuration created by the self-contact of a meander? In listing the possible configurations of entry and exit, we use a label that indicates where the two transits enter and exit: right or left, top or bottom. For most irreducible configurations, the requirements of noncrossing and nonoverlap fully determine the two transit curves, hence require a determined special way of erasing unused bonds and separating sites where there is contact. For the configuration labeled LBRT, the only sensible procedure is to perform site separation: with the same probabilities $s/2$, the center loop is attributed to one short transit or the other, and with the probability $1-s$, the center loop is set loose. For the configuration labeled LTLB, bond separation imposes itself.

The TMF is now found to be the 4×4 matrix

	(DT)	(PT)	(NS)	(OS)
(DT)	$(12-2s)/16$	$(1-s)/32$	$(4-4s)/16$	0
(PT)	$25/16$	$11/16$	$4s/16$	0
(NS)	$3/16$	$(14-s)/16$	$(32-12s)/16$	$24/16$
(OS)	$18/16$	$9/16$	$12/16$	$20/16$

For $s=1$, one finds $\lambda_1 = 2.96$, hence $D \sim 1.5656$. For $s=0$, we fall back on $D \sim 1.2886$. For intermediate s, the D ranges between the above values. Past experience[5] suggests that D is a near-linear function of the parameter s.

The expected number of self-contacts "born" in one generation from an existing self-contacting configuration is <1 on the average. According to a well-known chapter of probability theory, which was already used in Section 1.3.3, a population ruled by a birth process in which there is less than one offspring on the average will almost surely die off. The proofs in this theory do not apply here, because (as mentioned in Section 1.3.3) the births fail to be statistically independent. Nevertheless, the validity of its conclusion seems a safe conjecture. This is important, because it implies that every self-contact will eventually be separated away. On the other hand, site separation in the (NS) configurations keeps seeding new self-contacts. The number of self-contacts, per unit of length of the broken line approximation, remains constant on the average. When physical limitations force us to stop the interpolation, we are left with a curve rich in self-contacts. But they are mostly very local, and do not matter. In the limit squig interval the self-contacts can only occur between infinitesimally distant points.

1.4.4. Nonrandom Push Out or Pull in Separation

Suppose that one can interpret each squig interval as being a portion of a bigger loop that has been seeded at some earlier stage. The following separation processes can be defined: when a path meets a loop in a squig loop structure, it is instructed to bear toward the inside (or the outside) of the bigger loop. We still have some control over which paths are allowed to be considered. The choice concerns paths that create a double transit: they can be forbidden, allowed, or allowed with some prescribed probability.

1.5. SPATIAL SQUIG SHEETS AND SQUIG INTERVALS ON A CUBIC LATTICE

Squing intervals have not one, but two, distinct topological generalizations in the space $\mathbb{R}^3$, and $E-1$ distinct generalizations in the space $\mathbb{R}^E$. The reason is that separation in the plane sets up intervals that cannot be crossed, but separation in $\mathbb{R}^3$ may set up either intervals or squares that cannot be crossed. The nature of the separator determines whether the squig is a surface or is a curve. This topic is best discussed in terms of interpolation.

1.5.1. Squig Sheets on a Cubic Lattice of Base $b = 2$ in the Three-Dimensional Space $\mathbb{R}^3$

To generalize the notion of squig interval into that of a squig sheet, we must go through another exercise in definition. Section 1.3.3 had defined "interval" as a curve that is a bicontinuous map of $[0, 1]$, and is connected and minimal. Similarly, let us define a sheet as a bicontinuous map of the closed unit square. A sheet is a surface that is "contractible" and "minimal." Contractibility means that any portion of this surface can be deformed continuously while remaining part of it, until it reduces to any prescribed point on it, and minimality means that removal of any portion (not intersecting the curbe $\mathscr{C}$ that is the map of the boundary of our square) necessarily destroys contractibility.

Recall furthermore that a squig interval can be interpreted as the limit *either* of a sequence of "ribbons" whose widths decrease to zero while detail is being added, *or* of a sequence of broken lines forming the ribbon's backbones. Similarly, a squig sheet can be interpreted as the limit *either* of a sequence of "comforters" whose thickness decreases while detail is being added, *or* of a sequence of surfaces that form the comforter's backbone. These surfaces are made up of squares and can be called "sigma squares" or "σ-squares."

To implement squig sheets, I require their intersections by the faces of the cubes in the lattice to be squig intervals on a square lattice, as defined in Sections 1.4.2, 1.4.3, or 1.4.4. One can imagine that a loop made of M squig intervals is drawn on the unit cube's surface according to either of the rules in Section 1.4, then is spanned by a sheet, according to either one of the separation rules described below. But of course the operations of drawing the loop on the cube's surface, and of spanning the loop are really carried out in parallel.

To obtain a first approximation of our sheet by squares, we divide our cube of side one into eight subcubes of side 1/2. To each of the unit cube's

12 edges corresponds a square orthogonal to it at its midpoint; this square is one of the subinterfaces, i.e., of the interfaces between subcubes. The initial conditions determine which M edges of the unit cube will be crossed by the sheet. Under these conditions, one can approximate the sheet by M of the subinterfaces.

The next construction stage begins by performing decimations. On each of the 12 edges which our approximate sheet intersects, decimation pinpoints a half-edge that will be crossed. Then one replaces each subinterface by a collection of sub^2 interfaces of side 1/4. There are 96 such sub^2 interfaces and the first task is to sort them into two bins. The first bin contains the 72 sub^2 interfaces that touch the surface of our unit cube: each of its 12 edges is touched by two such sub^2 interfaces, and each of its six faces is touched by eight additional ones. The second bin contains the 24 remaining sub^2 interfaces. They bound a cube of side 1/2, whose vertices are centers of eight subcubes in our lattice and which is the equivalent of the loop in the plane; it will be called the "balloon." The idea, then, is to first refine the trace of our sheet on each of the faces of the unit cube, from being formed by segments of length 1/2 to being formed of segments of length 1/4. Then, each sub^2 interface that includes one of the latter segments is included in the squig sheet's second approximation. In exceptional configurations, these interfaces may form a minimal contractible surface. But in general these interfaces add up to a "rim" broken by a hole whose boundary (a broken line) we shall call "belt." The belt is drawn on the surface of the balloon separating it into two "half-balloons." The sum of the rim and the punctured balloon is always a contractible surface but it is not a sheet. The idea of generalized separation is to combine the rim and one of the two half-balloons into a sheet.

1.5.2. Pure Bond Separation Does Not Generalize to Three Dimensions

The spatial form of bond separation consists in puncturing the above cubic balloon in the center of one of its six faces, chosen at random. One removes the punctured face. Then one tests the four sides of this face. When a side does *not* belong to the belt, the "middle" face beyond that side is removed. Lastly, but only if at least one of the "middle" faces has been removed, one must test the sixth face of the balloon (farthest from the puncture). When this farthest face touches one of the removed faces along a side that does not belong to the belt, the farthest face is removed. This terminates the first stage of the process. The idea is that the separator in space is made of $M+1$ subedges of length 1/2, those segments being selected among the edges of the eight subcubes of the unit cubes in our lattice. Each of the first M separating segments is traced on a face of the unit cube, and touches these faces' centers. The last separating segment radiates from the center of the unit cube.

Unfortunately, the repetition of this process will eventually require an essential innovation. (This important fact was observed by Curtis McMullin.) Suppose that the rim intersects the surfaces of a lattice cube along a curve that transits only once by each of the six faces. The next stage of interpolation gives a positive probability to a configuration in which keeping *either* of the half-balloons will yield an approximate sheet that has either double points or double lines, and transits some subcubes *twice*. This last situation is not provided for by the rules of bond separation (Section 1.4.2), meaning that the configurations space of pure bond separation is not closed in three dimensions.

1.5.3. Acceptable Separation Processes. The Various Squig Sheets' Dimensionalities and Potential Applications

In order to have a closed configuration space, we must introduce loop separation rules that apply to double transits on the two-dimensional surface of the cubic cells, and balloon separation rules that apply to multiple transits in the three-dimensional cells. This can be done. For example, the push-out or pull-in methods of separation, in Section 1.4.4 generalize to sheets. Other examples have been examined. In each case, the TMF is huge. (See the Acknowledgment at the end of this paper.) It must be obtained by computer, and is not worth reprinting here. According to the degree of self-contact left in the approximations, we found values of D that range from 2.72 to 2.87.

Thus far, the theory of squigs has enjoyed a string of lucky breaks in the quality of the simplest squig models of the intervals, the trees and the clusters. As often stated, for example in FGN, p. 227, these breaks may go beyond simple coincidences, and may express some profound aspects of space that had not been previously tapped. In any event, this lucky streak may well continue: it would seem that the lower bound of the squig sheets' D is in the range of 8/3, which is a magic number in the study of turbulence (FGN, Chapter 30). A detailed study of the different Ds must be postponed to a more leisurely occasion.

1.5.4. Squig Intervals on a Cubic Lattice of Base $b = 2$ in Three Dimensional Space

In interpolation, the first step is to divide the unit cube into eight subcubes of side 1/2 and join the subcubes' centers. This yields 36 potential bonds falling into two bins. The 24 bonds in the first bin fall into six bunches of four bonds pointing out towards each of the six faces of the cube. These bonds are handled by decimation: of the four bonds in the interval's entry and exit faces, three are decimated and one is activated. The 12 potential bonds in the second bin form an elementary loop in space. The first step toward obtaining squig intervals is to separate this loop into a tree. This is achieved by preventing some of the square interfaces between subcubes from being crossed. Thus, the separators must be selected among the square interfaces. I find that one must separate five out of the 12 interfaces, but many configurations are unacceptable because they disconnect the loop. More generally, the configuration of the square separators must satisfy a number of constraints.

The total number of allowable configurations being high, a computer program was necessary to list them, to evaluate the TMF matrix, and to obtain the dimensionality. (See the Acknowledgment at the end of the paper.)

The leading eigenvalue was found to be $\lambda_1 = 3.7248$, which yields the fractal dimensionality of $D = 1.897$. The topic is far from exhausted.

1.5.5. Squig Intervals on a Cubic Lattice of Base $b > 2$. Is It Possible for Knots to Occur?

In cubic lattices with a value of $b > 2$, the squig constructions require multiple separators related by multiple constraints. My impression, but one that has not been implemented, is that one may arrange for knots to occur.

1.6. NONSYMMETRIC DECIMATION AND SEPARATION

1.6.1. Two Nonsymmetric Forms of Decimation

Let us return to the prototypical squig tree whose construction is recalled in Section 1.3.3. After four copies of $C(1)$ fit together to form $C(2)$, one of the two bonds between them is activated. As previously described, the activated bond is chosen at random, that is, with equal probabilities. Now we suppose that different probabilities p and $1 - p$ are adopted. Two specifications will be described. Together, they yield squig intervals that span the range between a straight segment and the Sierpinski gasked. In all cases, one can show (like in Ref. 9, p. 384) that the asymptotic squigs do not self-contact. I have as yet no concrete application for the asymmetric cases $p \neq 1/2$.

First specification. Start with a triangular lattice, orient one of the triangles arbitrarily, say, in the clockwise direction, and then orient the remaining triangles in a consistent fashion: the three that touch the original triangle by a side become oriented counterclockwise, and so on. Then the probability p is attached to activation of those potential bonds that lie to the left in clockwise cells and to the right in counterclockwise cells. Intuitive renormalization suggests, and Peyrière[7] proves, that the resulting squig intervals have the dimensionality $D = \log_2[3 - 2p(1-p)]$. Symmetric decimation, $p = 1/2$, yields the familiar $\log_2 2.5 = 1.3219$, a lower limit. Increasing asymmetry attributes increasing probability to large "eddies." Two maximally asymmetric decimations, $p = 1$ or $p = 0$, yield the Sierpinski gaskets constructed as Koch curves (FGN, p. 142) so that points of asymptotic contact are not allowed to be crossed. For them, D is the familiar $\log_2 3 = 1.5849$, an upper limit here.

Comment. The range from $\log_2 2.5$ to $\log_2 3$ is nearly identical to the range that was encountered and discussed in Section 1.4.3.

Second specification. When a tree is viewed as a branching river network, there is a consistent way of attaching the labels "left" and "right" to each side of each branch. In this setting, the squig intervals have the dimensionality $D = \log_2[3 - (p^2 \times (1-p)^2)]$. Its value lies between 1 and $\log_2 2.5 \sim 1.3219$.

The distribution of crossing points and the Besicovitch measure. Construct a squig interval by interpolation on a triangular lattice of base $b = 2$, starting with a unit triangle denoted as ABC. Suppose that all that is known initially is that the side through which the interval enters this triangle is AB. Successive stages of construction restricts the entry point to portions of AB of length 2^{-1}, 2^{-2}, 2^{-3}... When the construction is symmetric, $p = 1/2$, the position of the asymptotic entry point on AB is distributed uniformly. But in all the $p \neq 1/2$ cases, the positions of the asymptotic entry point follows a variant of a basic fractal measure, which I call B measure, in honor of Besicovitch. It is described in FGN, pp. 377 and 378 and in the Edinburgh paper, Section 3.2.1.

1.6.2. Nonsymmetric Substitutes for Separation

Suppose that the unseparated squig contains more than one interval from A to B. A variety of rules could be used as a substitute for separation. One may, at each stage of construction, decide that each time one reaches a fork, one shall systematically bear to the left. Alternatively, when one works on a triangular lattice, one can decide that when the $(k-1)$th approximation was bearing left, the loop in the kth approximation should be taken counter-clockwise.

1.7. EXAMPLES OF SQUIGS CONSTRUCTED ON UNCONVENTIONAL LATTICES

The tiling base b of a pertiling lattice is defined as the linear size of a supercell divided by that of a cell. In triangular or square tiling of the plane, or in cubic tilings of higher Euclidean spaces $\mathbb{R}^E$, the base is an integer b, and the number of cells in a supercell is b^E. The present section concerns several examples of more general lattices in which b^E or more generally b^D is an integer, but b itself need not be one. The first is an alternate rectangular lattice in the plane ($E=2$), whose base is the square root of an arbitrary integer $N \geqslant 2$. The second is a lattice in the plane ($E=2$), whose cells are bounded by fractal curves, and $b=\sqrt{5}$. The third example is a lattice that is traced on a fractal surface of dimension $D=\log_2 6>2$; here, $b=2$ but the number of cells in a supercell is not 2^2 but $2^D=6$. These examples were selected to illustrate the basic operations concerning squigs, and they are fairly independent of each other.

1.7.1. Pure Decimation: Squig Trees and Intervals on an Alternating Rectangular Lattice

Let $b=\sqrt{N}$, where N is an integer. The basic cell is a "generating rectangle" of length $\sqrt{N}$ and width 1. When N such rectangles are positioned horizontally and stacked on top of each other, they form a vertically positioned rectangle of length N and width $\sqrt{N}$. Next, N such rectangles are positioned side to side; together, they form a horizontally positioned rectangle of length $N\sqrt{N}$ and width N. The same process continues, alternating between horizontal rectangular rows and vertical rectangular columns.

Within each of the initial columns, there are at most $N-1$ potential bonds; let them *all* be activated. Within each second stage rectangle, the bonds subdivide into $N-1$ columns, each of which contains N potential bonds; let *one* bond be activated in each column, while the remaining $N-1$ bonds are "decimated." As the extrapolation continues, our process creates increasingly large trees, then binds them in rows or columns of N, etc... Asymptotically, one creates a plane-filling tree.

Squig intervals. In the above squig tree, a squig interval can traverse a cell in either of three ways, to be denoted as (SS), (LL), and (SL), meaning "short side to short side," etc. The squig interval's expected TMF takes the following form:

	(LL)	(SS)	(SL)
(LL)	$(N-1)(N-2)/3N$	N	$(N-1)/2$
(SS)	$1/N$	0	0
(SL)	$2(1-1/N)$	0	1

Again, the largest eigenvalue being denoted by λ_1, experience with other fractals,[6] and especially with squigs,[5] suggests that this squig interval's dimensionality is $\log \lambda_1/\log \sqrt{N}$. When $N = 2$, $\lambda_1 \sim 1.5514$, hence $D \sim 1.2671$, which is in the ballpark of $D \sim 4/3$ though on the small side. When $N \to \infty$, it is easy to see that $\lambda_1 \sim N/3$, hence $D \sim \log(N/3)/\log \sqrt{N} \sim 2 - 2/\log_3 N$. As it should, this D converges to 2, but it does so extremely slowly.

Generalization to the space $\mathbb{R}^3$. It uses tiles whose sides are in the ratios 1, $N^{1/3}$, and $N^{2/3}$. The squig interval's expected TMF is now a 6×6 matrix; I have not tried to write is down.

1.7.2. Pure Decimation: Squig Trees and Intervals on a Fractal Lattice of Base $\sqrt{5}$

This lattice is most conveniently described by starting with interpolation. The initiator is a unit square, and the generator is a lower case "*h*" whose four strokes are of equal length and make 90° angles. After one iteration, one has a fat "plus" sign made up of five squares of side $1/\sqrt{5}$: a central square and four squares stuck on the central square's sides. In the second iteration, each of these squares is replaced by five squares of side $(1/\sqrt{5})^2$. Asymptotically, one obtains the "quartet flake," namely, a domain shaped like a fuzzy fat plus sign and bounded by the fractal "quartet curve" shown in FGN, p. 49 (where it is covered by a peculiar Peano hatching). In the process, each side of the initiator square is first replaced by part of the boundary of the plus sign, namely, a *Z* whose three strokes are equal and make 90° angles; then by nine strokes, and asymptotically by one fourth of the fractal quartet curve.

The quartet flake having then been defined, we can take it as a basic cell, and proceed with an extrapolative construction, while thinking of the basic cell as a square whose sides have been "decorated." In a supercell made of five cells, there is a bond between the central cell and each of the remaining four, making a thin plus sign. consider a (super)^2 cell made of five supercells; between two neighboring thin plus signs, there are three potential bonds: two are decimated and one is activated. Next, we have nine potential bonds: eight are decimated and one is activated. The result is a plane-filling tree.

Squig intervals in the above squig trees. The interval can cross a cell in either of two directions: between neighboring sides (NS), or between opposite sides (OS). Therefore, the expected TMF is now the following 2×2 matrix:

	(NS)	(OS)
(NS)	15/9	16/9
(OS)	8/9	11/9

The largest real eigenvalue is $\lambda_1 = (13 + \sqrt{206})/9 = 3.039$, hence $D = \log \lambda_1 / \log \sqrt{5} = 1.381$.

1.7.3. Decimation and Separation: Squig Trees and Intervals on a Lattice Drawn on a Skew Fractal Surface

The imbedding surface considered here is already described in FGN, p. 139, where it is called Koch pyramid (and its secrets are unveiled). It is best constructed by interpolation; it will be understood that in this description all the triangles are equilateral. To define the generator of a fractal surface in 3-space, start with a triangle of side 1, divide in into four subtriangles of side 2^{-1}, build upon the middle triangle a regular tetrahedron, and erase the middle subtriangle of the original triangle. The resulting generator replaces one triangle by $N = 6$ triangles of base $b = 2$; hence the surface is of dimensionality $\log_2 6 = 2.5849$ $(= 1 + \text{the dimensionality of the Sierpinski gasket})$.

Now put together six tiles of the above surface to make up a supertile. Joining the centers of the triangles that bound the six tiles yields a graph made of one multiconnected ring and three dangling bonds. Continued extrapolation yields a fractal cluster that fills the Koch pyramid. If one wishes to pare this cluster down to a tree, the most natural process is to separate each ring somewhere.

Squig intervals in the above squig trees. While squares and rectangles can be traversed in three or two different ways (respectively), triangles can only be traversed in one way. Hence, the TMF reduces to a 1×1 matrix. Using the argument which the Edinburgh paper, Section 1.1.2 applied to triangular lattices of base $b = 3$, we find that the intervals in a randomly separated construction have the dimensionality $D = \log_2(14/4) = 1.8073$.

Observe that this value is much higher than the dimensionality given by the Flory formula for self-avoiding walks, $S_{SAW} = 2 - (1/3)(4 - D_{embedding})$.

Shortest and longest intervals. Now suppose that the rings are not separated, and consider the shortest paths between two points in such rings. It can be seen that their dimensionality is $\log_2(13/4) = 17004$. The longest intervals' dimensionality is $\log_2(16/4) = 2$.

1.7.4. Decimation and Multiple Separations: Squig Trees and Intervals on a Modified Kagome Lattice Linked to Snowflake Fractals

In the preceding example, a squig tree is contructed from a squig graph whose first stage includes a single ring. Therefore, a single separation suffices. In other examples, however, the first stage graph includes several rings. Multiple separations are then necessary and the novel fact is that they cannot be selected independently of each other.

For example, let us begin with a kagomé lattice (*kagomé* means *basket* in Japanese, hence takes a lower case initial). It is a mixture of hexagonal and triangular cells. Pick one of the hexagons as origin, and merge it with its six neighboring triangles to form a starred hexagon (Star of David), the first stage of the construction of a Koch snowflake. Do the same with the six next nearest neighbor hexagons, that is, the hexagons that lie just beyond the six that share a vertex with the original hexagon. Then repeat the same merging operation with each of the six next nearest neighbor hexagons.

The result, to be called modified kagomé lattice, is a mixture of starred and convex hexagons. In this lattice, a supercell is defined as a star, plus (a) the six hexagons that touch it by two sides, and (b) the six stars that touch it by one vertex. The supercell's shape is the second stage of the construction of the Koch snowflake. Between the centers of neighboring cells within a supercell, there are 18 potential bonds; when all are activated, the resulting graph includes many loops; in particular, it includes six minimal loops of six potential bonds, two of them going through the center of the middle star, and a loop made of six potential bonds that goes around the graph. It is easy to see that there are several ways of achieving full separation (hence, of obtaining a tree), by breaking six of the potential bonds. One possible implementation consists in breaking one bond in each of the minimal loops, with the following constraints: each broken bond is credited to only one loop, and breaking all six bonds that start at the origin is not allowed.

1.8. SQUIGS AS UNCONVENTIONAL GEOMETRIC BIRTH PROCESSES. *M* SYSTEMS. REFERENCE TO PEYRIÈRE'S PROOF THAT RENORMALIZATION ARGUMENTS USING TMF ARE RIGOROUS

Let us return to the simplest squig interval on a triangular lattice of base $b = 2$. Section 1.3.3 introduced birth processes and observed that the numbers of daughters born to neighboring mothers fail to be independent, hence the theory of "simple" birth processes fails to apply. An intuitive renormalization argument followed by numerical tests suggested to me that this should not matter. Peyrière[7] proved this fact rigorously and placed it in the context of a very interesting general theory. Recall that a triangle is right (left) handed if the broken line approximation turns right (left) in this triangle. The geometrical birth process underlying squig intervals can be expressed by saying that the numbers and the "handedness" of each mother's progeny is determined by whether the mother is left or right handed, and whether the separations on both sides of this mother are to the left or to the right. As a result, the passage from one to the next consists in the following three steps: take a word of N letters, the alphabet being reduced to L and R; next add a random letter, either R or L, between any two neighboring letters, and also at both ends, thus obtaining a word of $2N + 1$ letters, and finally apply to each sequence new-old-new letters the following curious rules of "genetics":

$$L(L)L \to L, \qquad L(L)R \to (RLL), \qquad L(R)L \to (RLR), \qquad L(R)R \to (RRL)$$

$$R(R)R \to R, \qquad R(R)L \to (LRR), \qquad R(L)R \to (LRL) \qquad R(L)L \to (LLR)$$

Peyrière[7] has generalized this pseudo-genetics into a broader notion of M system. (They generalize the "L systems" named after A. Lindenmeyer.) See also Ref. 9.

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